Population dynamics models and MCMC

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Population dynamics models and MCMC

Purpose: Present the process of developing and applying a simple population dynamics model where uncertainty in the parameters is analysed using Bayesian Markov Chain Monte Carlo (MCMC) methods.

Typical fisheries stock assessment problem

Given:

- Time series of removals from population
- Index of relative abundance (catch per unit of effort, CPUE)
- Desire to set limits on catch to ensure sustainability
- Interest in uncertainty around estimates

Common approach:

- Develop population dynamics model
- Measure fit of model to data given particular parameter values
- Use Bayesian methods to characterize uncertainty

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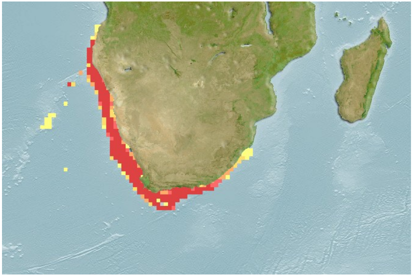
Example data: South African hake



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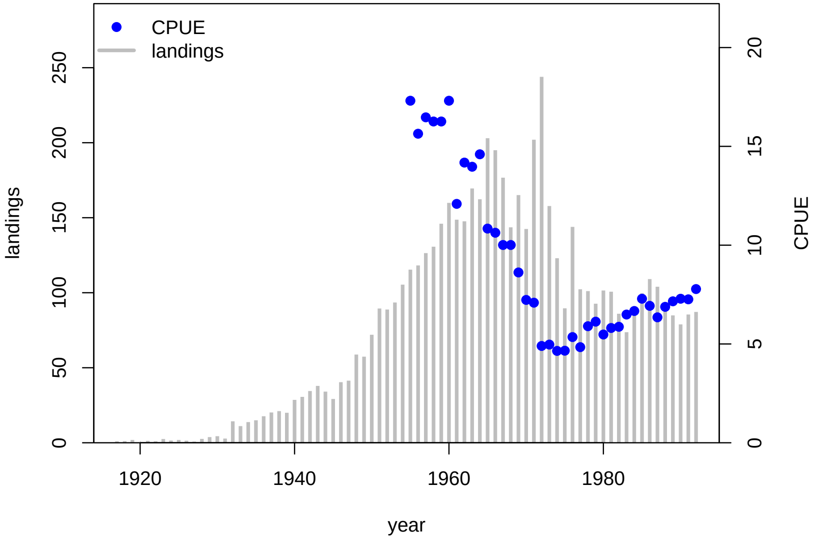


Biologically unreasonable population dynamics model: the rabbits of Leonardo of Pisa, a.k.a. Fibonacci[1]

- in the first month there is just one newly-born pair,
- new-born pairs become fertile from after their second month
- each month every fertile pair begets a new pair, and
- the rabbits never die

Extended model with catch and density dependence:

M = mature pairs, N = new-born pairs

$$M_{t+1} = M_t + N_t - C_t$$

$$N_{t+1} = M_t \cdot r \left(1 - \frac{M_t}{K}\right)$$

[1] as described at http://en.wikipedia.org/wiki/Fibonacci_number

Historical aside on the Fibonacci numbers[2]

Fibonacci numbers appeared under the name *mātrāmeru* (mountain of cadence), in the work of the Sanskrit grammarian Pingala (*Chandah-shāstra*, the Art of Prosody, 450 or 200 BC). The Indian mathematician Virahanka (6th century AD) showed how the Fibonacci sequence arose in the analysis of metres with long and short syllables.

Sanskrit vowel sounds can be long (L) or short (S), and Virahanka's analysis, which came to be known as *mātrā-vṛtta*, wishes to compute how many metres (*mātrās*) of a given overall length can be composed of these syllables. If the long syllable is twice as long as the short, the solutions are:

1 mora: S (1 pattern)
2 morae: SS; L (2 patterns)
3 morae: SSS, SL; LS (3 patterns)
4 morae: SSSS, SSL, SLS; LSS, LL (5 patterns)
5 morae: SSSSS, SSSL, SSLS, SLSS, SLL; LSSS, LSL, LLS (8 patterns)

[2]source: http://en.wikipedia.org/wiki/Fibonacci_number

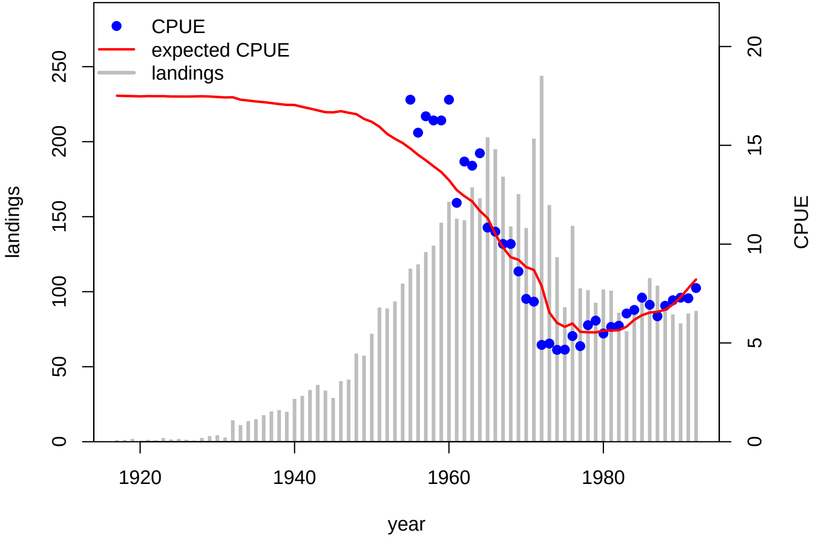
Markov chain Monte Carlo & the Metropolis-Hastings Algorithm

- used when other sampling methods fail
- samples from chain are effectively i.i.d. samples from distribution (when chain is long enough, and sufficiently thinned)
- easy to implement, but sometimes slow to run, difficult to assess convergence.
- Metropolis-Hastings is particular implementation of MCMC:
- from each value $\mathbf{x}^\mathbf{t}$ in chain, generate new value $\mathbf{x}'$ using appropriate jump function
- if new value is better, accept as next element in chain
- if new value is worse, accept with probability $P(\mathbf{x}')/P(\mathbf{x}^\mathbf{t})$

Fit of Fibonacci rabbit model to South African hake

$K = 1524$, $r = 0.3522$,

but how do we construct confidence intervals?



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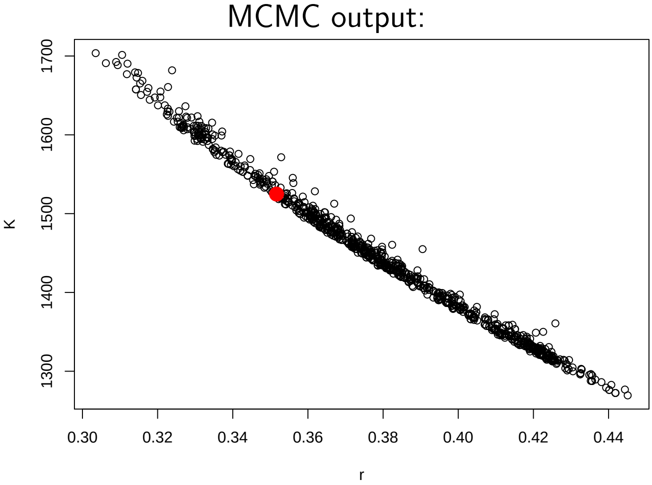

Calculating Maximum Sustainable Yield

$$M_{t+1} = M_t + N_t - C_t$$
$$N_{t+1} = M_t \cdot r \left(1 - \frac{M_t}{K}\right)$$

As long as $C_t \leq N_t$, harvest with be sustainable.
Maximum sustainable yield occurs when N is maximized, or

$$\begin{aligned}
\frac{\partial N}{\partial M} &= 0 \\
\frac{\partial}{\partial M} M \cdot r \left(1 - \frac{M}{K}\right) &= 0 \\
1 - \frac{2M}{K} &= 0 \\
M &= K/2
\end{aligned}$$

substitution gives

$$MSY = N_{MSY} = rK/4$$

Reparameterizing the extended Fibonacci model

Given

$$MSY = rK/4,$$

we can treat MSY as a parameter and treat K as a derived quantity

$$K = 4MSY/r.$$