Introduction to stochastic processes

ed: Eli Gurarie

QERM 598 - Lecture 6

February 14, 2008

A Stochastic Processes Is:

- Any process in which outcomes in some variable (usually time, sometimes space, sometimes something else) are uncertain and best modelled probabilistically.
- **stochastic** is to **deterministic** as **random variable** is to **number**
- Biggest difference from what we've done so far: **Dependent Data**.

Examples include:

Just about everything.

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In ecology/biology, just about everything includes:

- Weather/Climate
- Population biology
  - Birth/death/reproduction/mortality
  - Migrations and movements
- Evolution
  - Population genetics (Mutation/Selection/Drift)
  - Gene sequences
- Epidemiology
  - Disease spread within a population (SIR models)
  - Disease spread within an organism
  - Development of resistance
- Tools for assessing models and estimating parameters
  - MCMC (Markov Chain Monte Carlo)
  - Simulated annealing
- and much, much more

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Just about anything ALSO includes:

Your life!

- You drop out of school to make lots of money in the stock market
- You lose all your money gambling (Bernoulli and Bernoulli 1713)
- You eventually do or don't find your way home from some unknown establishment you've chosed to drink your sorrows away in (Pearson 1905)

These are all Classic problems in stochastic processes!

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Historical aside on stochastic processes

Andrei Andreevich Markov (1856-1922) was a brilliant Russian mathematician who refused to believe that the Central Limit Theorem only applies to independent data, and consequently came up with the most widely used formalism and much of the theory for stochastic processes.

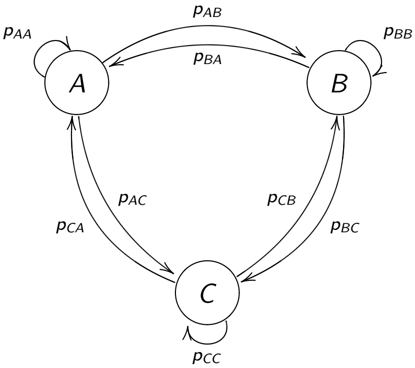
Passionate about math pedagogy, he was a strong proponent of problem-solving over seminar-style lectures. A political activist, he refused tsarist honors, requested that he be excommunicated from the Russian Orthodox Church out of solidarity with the recently excommunicated **Leo Tolstoy**, publicly renounced his "membership in the electorate" when Parliament was dissolved, and eventually left his teaching post when the government demanded that teachers spy on students with socialist sentiments.

He said this of his most famous English colleague: *"I can judge all work only from a strictly mathematical point of view and from this viewpoint it is clear to me that ... Pearson has done nothing of any note."*[1]

[1] from: Basharin et al. (2005) The Life and Work of A.A. Markov

Discrete state transitions

Consider $\mathbf{X} = \{X_1, X_2, X_3, ..., X_n\}$ is in some discrete state space $\mathcal{E}$ (here: A, B, C) with fixed probabilities of transitioning from one state to another:



Sample sequence: $\mathbf{X} = CCCBBCACCBABCBA...$

This object is called a **Markov** chain.

Some definitions

$\mathbf{X}_n$ has the **Markov Property** if:

$$\Pr\{X_n = x_n | X_1 = x_1, \cdots, X_{n-1} = x_{n-1}\} = \Pr\{X_n = x_n | X_{n-1} = x_{n-1}\}$$

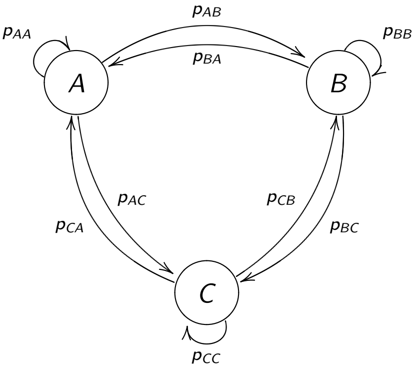
for all n in $x_1, \cdots, x_n$.

In other words, any system whose future depends *only* on the present and not on the past has the *Markov Property* and any Markovian $\mathbf{X}_n$ is Markovian is called a **Markov Chain**.

The $p_{ij}(t)$'s of a Markov chain are transition probabilities. If $p_{ij}(t)$'s are time invariant, $(p_{ij}(t) = p_{ij})$, the chain is called **time homogeneous** or is said to have **stationary transition probabilities**.

Discrete state transitions

We express this process in terms of a **Probability Transition** matrix:



$$\mathbf{M} = \begin{array}{c|ccc|} \textit{from}: \backslash^{\textit{to}:} & \text{A} & \text{B} & \text{C} \\ \hline \text{A} & p_{AA} & p_{AB} & p_{AC} \\ \text{B} & p_{BA} & p_{BB} & p_{BC} \\ \text{C} & p_{CA} & p_{BC} & p_{CC} \end{array}$$

Such that:

$$M_{ij} = \Pr\left(X_{t+1} = j | X_t = i\right) = p_{ij} \tag{1}$$

Note that:

$$\sum_{j=1}^{n} p_{ij} = 1 \ldots \text{BUT} \ldots \sum_{i=1}^{n} p_{ij} \neq 1 \tag{2}$$

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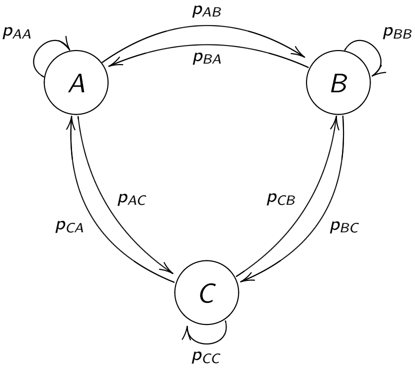

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Such that:

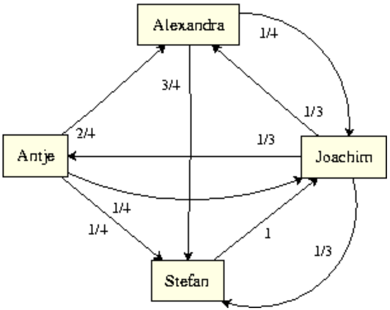
$$\Pr\left(X_{t+1} = j = \sum_{i=1}^{N} M_{ij} \Pr\left(X_t = i\right)\right. \tag{3}$$

Which can be conveniently rewritten in matrix notation as:

$$\pi_{t+1} = \mathbf{M} \times \pi_t \tag{4}$$

Where π_t is the distribution of the system over all states at time t.

Example 1: German children play catch[2]



$$\mathbf{M} = \begin{array}{c|cccc|} & An & Al & Jo & St \\ \hline An & 0 & 3/4 & 0 & 1/4 \\ Al & 0 & 0 & 1/4 & 3/4 \\ Jo & 1/3 & 1/3 & 0 & 1/3 \\ St & 0 & 0 & 1 & 0 \end{array}$$

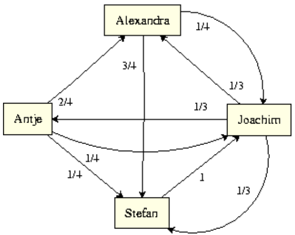
Let's give the ball to Antje, and see what happens:

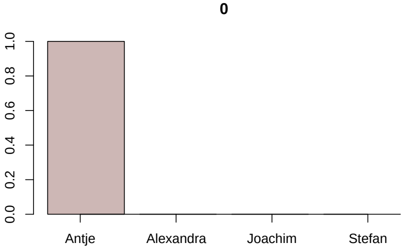
An - Al - St - Jo - St - Jo - Al - St - Jo - St - Jo - Al - Jo - St - Jo - St - Jo - Al - Jo - An - Al - St - Jo - Al - Jo - Al - St - Jo - Al - St - Jo - An - St - Jo - St - Jo - St - Jo - An - St - Jo - An - Al - Jo - St - Jo - An - Al - St - Jo - St - Jo - An - Al - Jo - Al - St - Jo - Al - St - Jo - An - Al - St - Jo - Al - St - Jo - etc ...

This is called a **realization** of a stochastic process.

[2] http://www.leda-tutorial.org/en/unofficial/Pictures/MarkovChain.png

Consider the process probabalistically

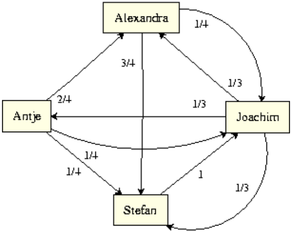


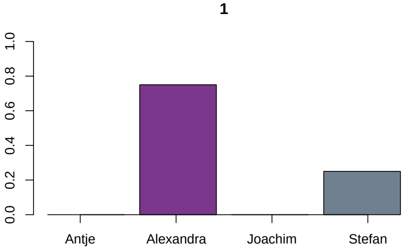


Give the ball to Antje again:

$$\pi_0 = (1, 0, 0, 0)$$

Consider the process probabalistically





Give the ball to Antje again:

$$\pi_0 = (1, 0, 0, 0)$$
$$\pi_1 = (0, 0.75, 0, 0.25)$$

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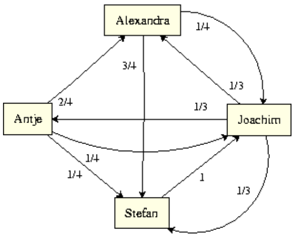

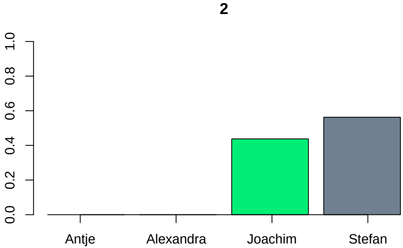

Give the ball to Antje again:

$$\pi_0 = (1, 0, 0, 0)$$
$$\pi_1 = (0, 0.75, 0, 0.25)$$
$$\pi_2 = (0, 0, 0.438, 0.562)$$

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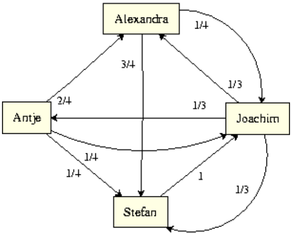

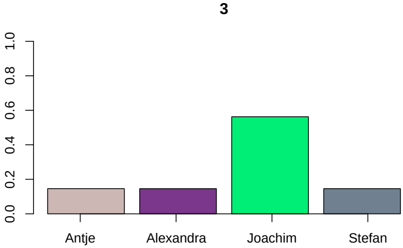

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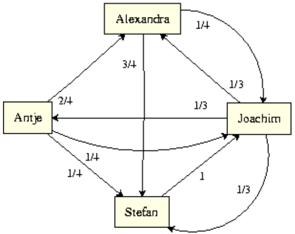

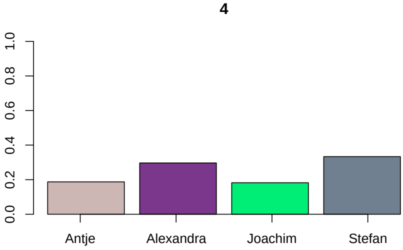

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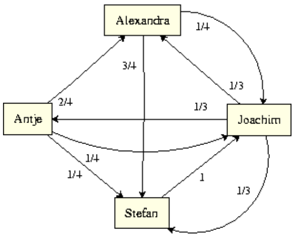

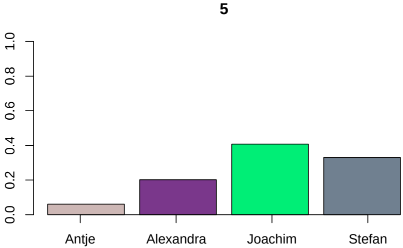

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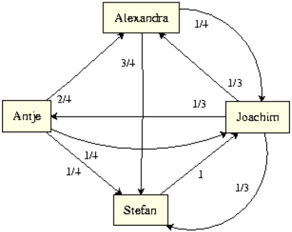

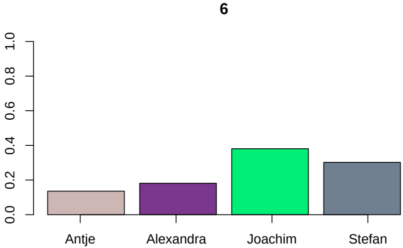

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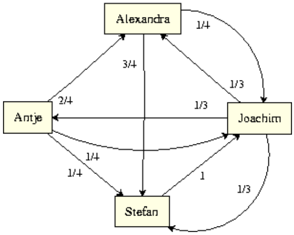

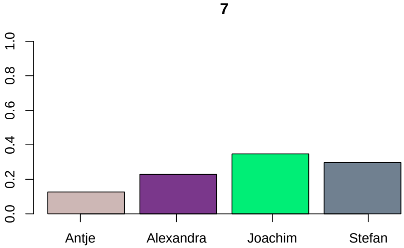

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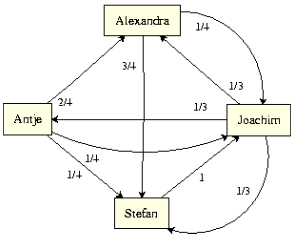

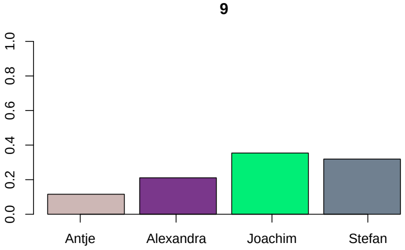

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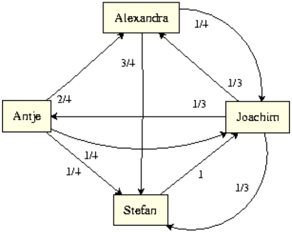

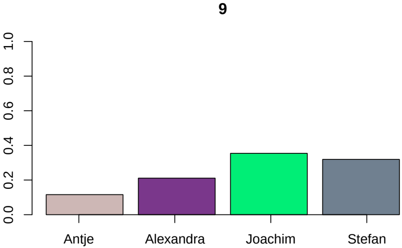

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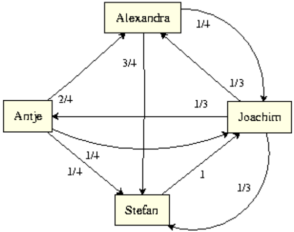

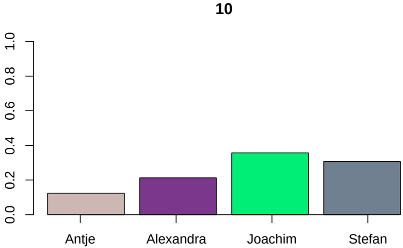

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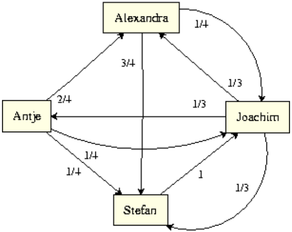

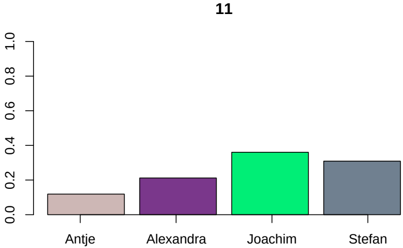

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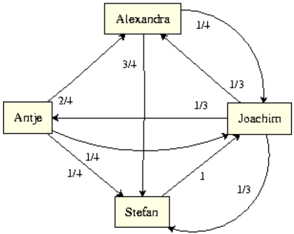
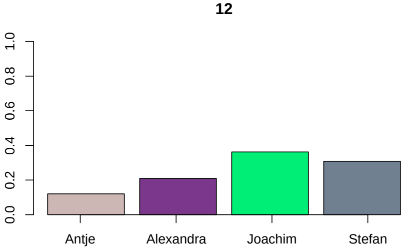
Consider the process probabalistically

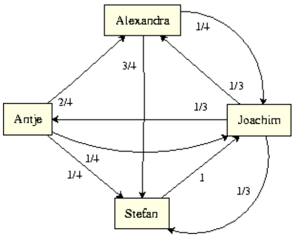


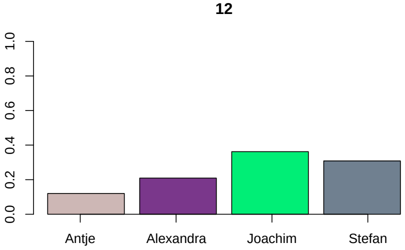


Give the ball to Antje again:

$$
\begin{aligned}
\pi_0 &= (1, 0, 0, 0) \\
\pi_1 &= (0, 0.75, 0, 0.25) \\
\pi_2 &= (0, 0, 0.438, 0.562) \\
\pi_3 &= (0.146, 0.146, 0.562, 0.146) \\
\pi_4 &= (0.188, 0.297, 0.182, 0.333) \\
\pi_5 &= (0.061, 0.201, 0.408, 0.33) \\
\pi_6 &= (0.136, 0.181, 0.381, 0.302) \\
\pi_7 &= (0.127, 0.229, 0.347, 0.297) \\
\pi_8 &= (0.116, 0.211, 0.354, 0.319) \\
\pi_9 &= (0.118, 0.205, 0.372, 0.305) \\
\pi_{10} &= (0.124, 0.212, 0.356, 0.307) \\
\pi_{11} &= (0.119, 0.212, 0.36, 0.309) \\
\pi_{12} &= (0.12, 0.209, 0.362, 0.309)
\end{aligned}
$$

Consider the process probabalistically





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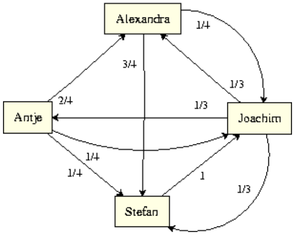
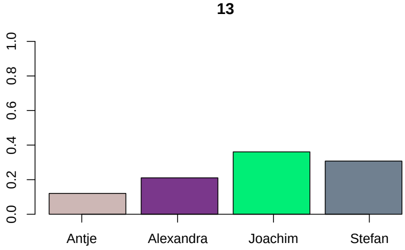
\pi_{13} &= (0.121, 0.211, 0.361, 0.308)
\end{aligned}
$$

Consider the process probabalistically





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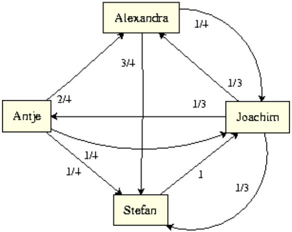
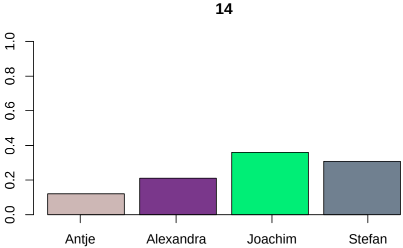
\pi_{13} &= (0.121, 0.211, 0.361, 0.308)
\end{aligned}
$$

Consider the process probabalistically





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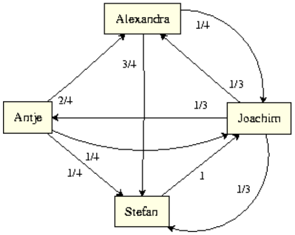
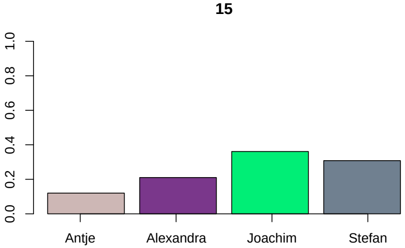
\pi_{13} &= (0.121, 0.211, 0.361, 0.308)\\
\pi_{14} &= (0.12, 0.211, 0.36, 0.309)
\end{aligned}
$$

Consider the process probabalistically





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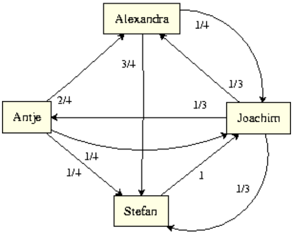
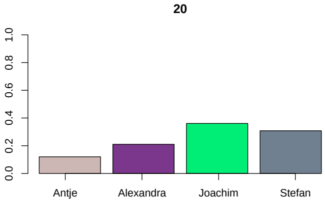
\pi_{12} &= (0.12, 0.209, 0.362, 0.309)\\
\pi_{13} &= (0.121, 0.211, 0.361, 0.308)\\
\pi_{14} &= (0.12, 0.211, 0.36, 0.309)\\
\pi_{15} &= (0.12, 0.21, 0.361, 0.308)
\end{aligned}
$$

The stationary state





The state: $\pi^* = (0.12, 0.21, 0.361, 0.308)$ is referred to as **stationary**. Note that

1. The name is a little bit misleading: the ball is not stationary, it is always moving around.
2. The state can be solved for mathematically:

$$\pi^* = \mathbf{M} \times \pi^* \tag{5}$$

   This is a straighforward linear algebra problem, and is usually easy to obtain (for Mathematica).

3. All states have a value between 0 and 1 and have finite probablity of being revisited forever and ever until the children's arms fall off. Such states are termed **recurrent**, **persistent** or **ergodic**.

Example 2: Genetic Drift

Consider a population of size N that features 2 **alleles** (A and B) at a given **locus**.

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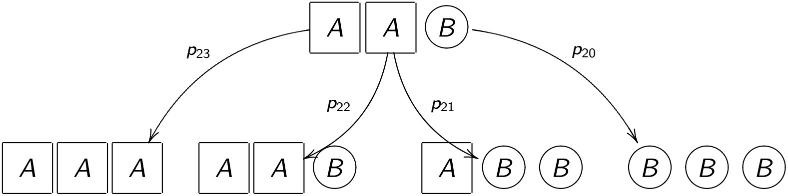
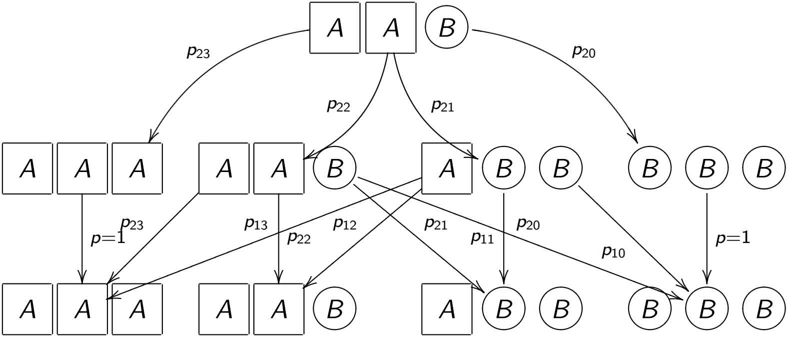


Example 2: Genetic Drift

Consider a population of size N that features 2 **alleles** (A and B) at a given **locus**.



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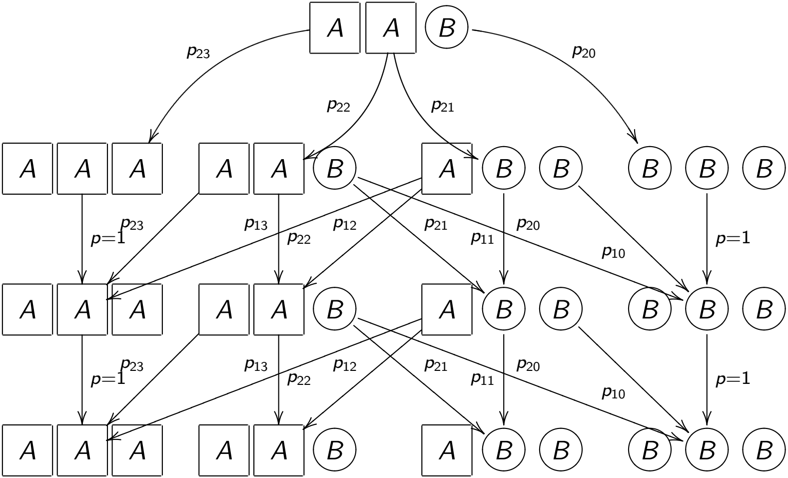
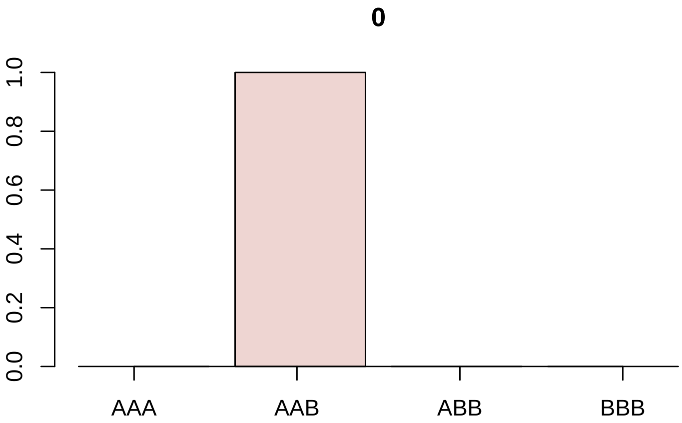


Genetic-Drift: Fisher-Wright Matrix

If the State X is defined as number of A alleles in the population, then:

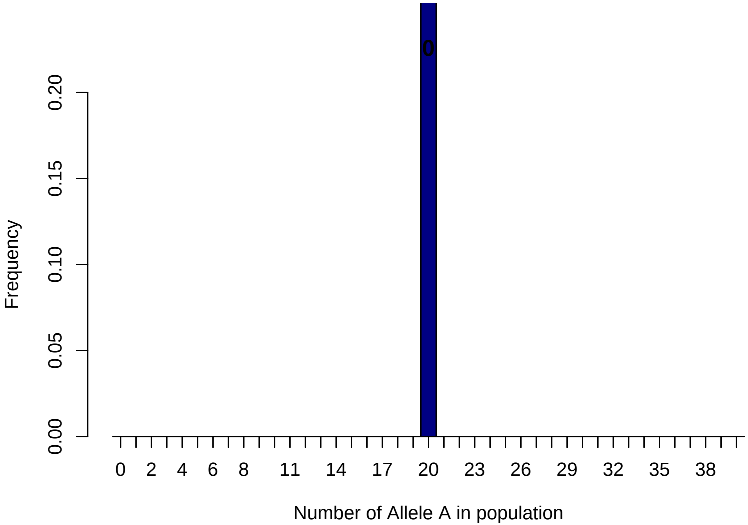
$$\mathbf{M} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ \left(\frac{2}{3}\right)^3 & 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 & 3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ \left(\frac{1}{3}\right)^3 & 3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 & 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{M} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.296 & 0.444 & 0.222 & 0.037 \\ 0.037 & 0.222 & 0.444 & 0.296 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix} \end{array}$$



click on image to move forward

N=40



click on image to start/pause animation

Fixation and transience

General Fisher-Wright matrix:

$$p_{ij} = Pr\{A_{t+1} = j | A_t = i\} = \binom{2N}{j} \left(\frac{i}{2N}\right)^j \left(1 - \frac{i}{2N}\right)^{2N-j} \quad (6)$$

Some properties of genetic drift:

- Always eventually fixates at 0 or N.
- Proportion of fixation depends on initial proportion of a given allele.
- Rate of fixation depends inversely on N
- Other states are called **transient** (contrasted with **recurrent**), because the process does not necessarily return to them.

The final moral:

- **Genetic drift** is a stochastic fluctuation in allele frequencies that leads inexorably to fixation for small populations, but is counteracted by **mutation** and **migration** for large populations.

Example 3: Blackjack

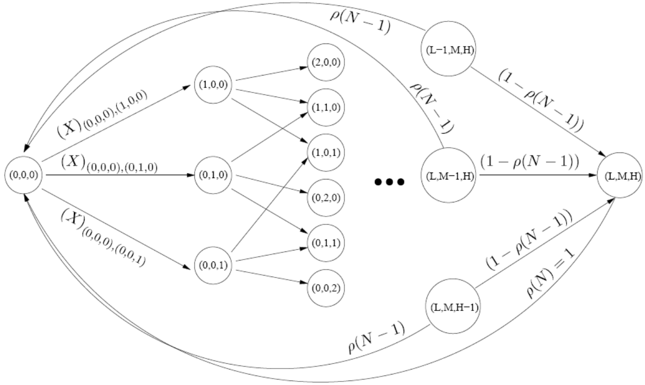


Figure 2: *Graphical depiction of full state space* Σ. *Each state represents an ordered triple* (L, M, H) *denoting the number of low, medium, and high cards that have been played from the shoe.*

Blackjack state-space analysis from

http://cmc.rice.edu/docs/docs/Wak2004Jul1AMarkovCha.pdf